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ABSTRACT

A central problem in signal processing and communications is to design signals that are compact both in time and frequency. Heisenberg's uncertainty principle states that a given function cannot be arbitrarily compact both in time and frequency, defining an “uncertainty” lower bound. Taking the variance as a measure of localization in time and frequency, Gaussian functions reach this bound for continuous-time signals. For sequences, however, this is not true; it is known that Heisenberg's bound is generally unachievable. For a chosen frequency variance, we formulate the search for “maximally compact sequences” as an exactly and efficiently solved convex optimization problem, thus providing a sharp uncertainty principle for sequences. Interestingly, the optimization formulation also reveals that maximally compact sequences are derived from Mathieu's harmonic cosine function of order zero. We further provide rational asymptotic expansions of this sharp uncertainty bound. We use the derived bounds as a benchmark to compare the compactness of well-known window functions with that of the optimal Mathieu's functions.

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1. Introduction

Suppose you are asked to design filters that are sharp in the frequency domain and at the same time compact in the time domain. The same problem is posed in designing sharp probing basis functions with compact frequency characteristics. In order to formulate these problems mathematically, we need to have a correct and universal definition of compactness and clarify what we mean by saying a signal is spread in time or frequency.

These notions are well defined and established for continuous-time signals [13,34] and their properties are studied thoroughly in the literature. For such signals, we can define the time and frequency characteristics of a signal as in Table 1. Note the connection of these definitions with the mean and variance of a probability distribution function $|x(t)|^2/\|x\|^2$. The value of Δ_t^2 is considered as the spread of the signal in the time

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Table 1
Time and frequency centers and spreads for a continuous time signal $x(t)$.

Domain	Center	Spread
Time	$\mu_t = \frac{1}{\ x\ ^2} \int_{t \in \mathbb{R}} t x(t) ^2 dt$	$\Delta_t^2 = \frac{1}{\ x\ ^2} \int_{t \in \mathbb{R}} (t - \mu_t)^2 x(t) ^2 dt$
Frequency	$\mu_{\omega_c} = \frac{1}{2\pi\ x\ ^2} \int_{\omega \in \mathbb{R}} \omega X(\omega) ^2 d\omega$	$\Delta_{\omega_c}^2 = \frac{1}{2\pi\ x\ ^2} \int_{\omega \in \mathbb{R}} (\omega - \mu_{\omega_c})^2 X(\omega) ^2 d\omega$

Table 2
Time and frequency centers and spreads for a discrete time signal x_n as extensions of [Table 1](#) [34].

Domain	Center	Spread
Time	$\mu_n = \frac{1}{\ x\ ^2} \sum_{k \in \mathbb{Z}} k x_k ^2$	$\Delta_n^2 = \frac{1}{\ x\ ^2} \sum_{k \in \mathbb{Z}} (k - \mu_n)^2 x_k ^2$
Frequency	$\mu_{\omega_\ell} = \frac{1}{2\pi\ x\ ^2} \int_{-\pi}^{\pi} \omega X(e^{j\omega}) ^2 d\omega$	$\Delta_{\omega_\ell}^2 = \frac{1}{2\pi\ x\ ^2} \int_{-\pi}^{\pi} (\omega - \mu_{\omega_\ell})^2 X(e^{j\omega}) ^2 d\omega$

domain while $\Delta_{\omega_c}^2$ represents its spread in the frequency domain. We say that a signal is compact in time (or frequency) if it has a small time (or frequency) spread.

The Heisenberg uncertainty principle [13,27,28] states that continuous-time signals cannot be arbitrarily compact in both domains. Specifically, for any $x(t) \in L^2(\mathbb{R})$,

$$\eta_c = \Delta_t^2 \Delta_{\omega_c}^2 \geq \frac{1}{4}, \tag{1}$$

where the lower bound is achieved for Gaussian signals of the form $x(t) = \gamma e^{-\alpha(t-t_0)^2 + j\omega_0 t}$, $\alpha > 0$ [9]. The subscript c stands for continuous-time definitions. We call η_c the *time–frequency spread* of x .

Although the continuous Heisenberg uncertainty principle is widely used in theory, in practice we often work with discrete-time signals (e.g. filters and wavelets). Thus, equivalent definitions for discrete-time sequences are needed in signal processing. In the next section we study two common definitions of center and spread available in the literature.

1.1. Uncertainty principles for sequences

An obvious and intuitive extension of the definitions in [Table 1](#) for discrete-time signals is presented in [Table 2](#), where

$$X(e^{j\omega}) = \sum_{k \in \mathbb{Z}} x_k e^{-j\omega k} \quad \omega \in \mathbb{R}, \tag{2}$$

is the discrete-time Fourier transform (DTFT) of x_n .

Using the definitions in [Table 2](#) [34], we can also state the Heisenberg uncertainty principle for discrete-time signals as

$$\eta_\ell = \Delta_n^2 \Delta_{\omega_\ell}^2 > \frac{1}{4}, \quad x_n \in \ell^2(\mathbb{Z}) \text{ with } X(e^{j\pi}) = 0, \tag{3}$$

where the subscript ℓ stands for *linear* in reference to the definition of the frequency spread. Note the extra assumption on the Fourier transform of the signal in (3). This assumption is necessary for the result to hold.

Example 1. Take $x_n = \delta_n + 7\delta_{n-1} + 2\delta_{n-2}$. It is easy to verify that $|X(e^{j\pi})| = 0.22 \neq 0$, which violates the condition $X(e^{j\pi}) = 0$. The linear time–frequency spread of this signal according to [Table 2](#) is $\eta_\ell = 0.159 < 1/4$.

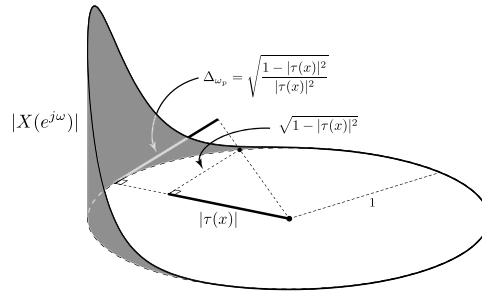


Fig. 1. $X(e^{j\omega})$ is a periodic function defined on the unit circle. The figure shows the correspondence between $\tau(x)$ and the periodic frequency spread of the signal.

In addition to the restriction on the Heisenberg uncertainty principle, the definitions in Table 2 do not capture the periodic nature of $X(e^{j\omega})$ for the frequency center and spread. In the search for more natural properties, we can adopt definitions for circular moments widely used in quantum mechanics [4] and directional statistics [18].

Definition 1. For a sequence $x_n, n \in \mathbb{Z}$, with a 2π -periodic DTFT, $X(e^{j\omega})$ as in (2), the *first trigonometric moment* is defined as [25,26]

$$\begin{aligned} \tau(x) &= \frac{1}{2\pi\|x\|^2} \int_{-\pi}^{\pi} e^{j\omega} |X(e^{j\omega})|^2 d\omega \\ &\stackrel{(a)}{=} \frac{1}{\|x\|^2} \sum_{k \in \mathbb{Z}} x_k x_{k+1}^* \end{aligned} \tag{4}$$

where (a) follows from Parseval’s equality.

The first trigonometric moment was originally defined for probability distributions on a circle. With proper normalization, this definition applies also to periodic functions.

Definition 2. Using (4), the *periodic frequency spread* is defined as [4]:

$$\Delta_{\omega_p}^2 = \frac{1 - |\tau(x)|^2}{|\tau(x)|^2} = \left| \frac{\|x\|^2}{\sum_{k \in \mathbb{Z}} x_k x_{k+1}^*} \right|^2 - 1, \tag{5}$$

where $\tau(x)$ is defined in (4). This definition makes only sense when $\tau(x) \neq 0$. If $\tau(x) = 0$, we set $\Delta_{\omega_p}^2 = \infty$. Fig. 1 illustrates pictorially how this definition corresponds to the first trigonometric moment of the periodic signal $X(e^{j\omega})$ (the figure is inspired from [8]).

The definition of Δ_n^2 remains unchanged as in Table 2. These definitions are summarized in Table 3. Using these definitions, Breitenberger [4] states the uncertainty relation for sequences as

$$\eta_p = \Delta_n^2 \Delta_{\omega_p}^2 \geq \frac{1}{4}, \quad \text{for } \|x\|_0 > 1. \tag{6}$$

The condition $\|x\|_0 > 1$ avoids the case $\Delta_n^2 = 0$, which would happen for $x_n = \gamma\delta_{n-n_0}$.

1.2. Contribution

In this paper, using the definitions in Table 3, we revisit the Heisenberg uncertainty principle for discrete-time signals. We address the fundamental yet unanswered question: If someone asks us to design a discrete

Table 3
Time and frequency centers and spreads for a discrete time signal x_n using circular moments, where $\tau(x)$ is defined in (4).

Domain	Center	Spread
Time	$\mu_n = \frac{1}{\ x\ ^2} \sum_{k \in \mathbb{Z}} k x_k ^2$	$\Delta_n^2 = \frac{1}{\ x\ ^2} \sum_{k \in \mathbb{Z}} (k - \mu_n)^2 x_k ^2$
Frequency	$\mu_{\omega_p} = 1 - \tau(x)$	$\Delta_{\omega_p}^2 = \frac{1 - \tau(x) ^2}{ \tau(x) ^2} = \left \frac{\ x\ ^2}{\sum_{k \in \mathbb{Z}} x_k x_{k+1}^*} \right ^2 - 1$

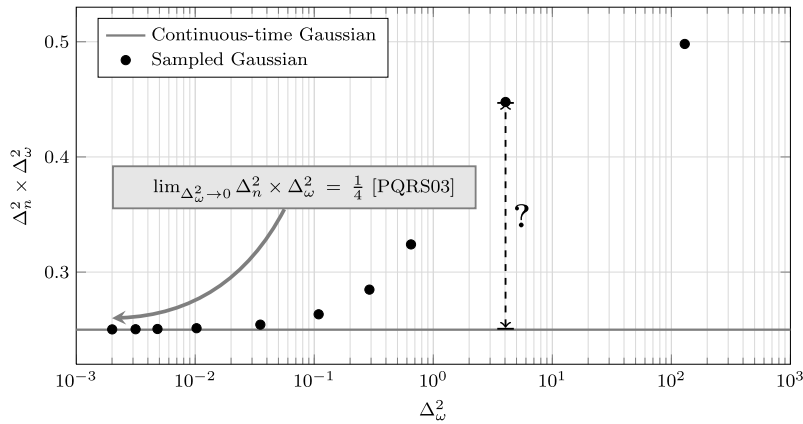


Fig. 2. Time–frequency spread of continuous vs. sampled Gaussians. The solid line shows the 1/4 Heisenberg bound which is achieved by continuous Gaussian signals (with definitions in Table 1), while the markers show the time–frequency spread (according to Table 3) of sampled Gaussian sequences. The question is if the gap between the two curves is inherited from the properties of sequences or sampled Gaussians are not optimal in discrete domain.

filter with a certain frequency spread ($\Delta_{\omega_p}^2$ fixed), can we return the sequence with minimal time spread Δ_n^2 ? In other words, the problem is to find the solution to

$$\begin{aligned} \Delta_{n,\text{opt}}^2 &= \underset{x_n}{\text{minimize}} && \Delta_n^2 \\ &\text{subject to} && \Delta_{\omega_p}^2 = \sigma^2(\text{fixed}). \end{aligned} \tag{7}$$

In order to provide an insight on the uncertainty principle in the discrete-time domain, we do a simple test. In Fig. 2 we show the time–frequency spread of continuous Gaussian signals by the solid line on 1/4. This is the Heisenberg’s uncertainty bound which is achievable by continuous-time Gaussians. Further, using the definitions in Table 3, we compute the time–frequency spread of sampled Gaussian sequences. According to Prestin et al. [26], the time–frequency spread tends to 1/4 as the frequency spread of Gaussians decreases. However, when the frequency spread is large, the values are far from the uncertainty bound. Two questions arise here; is the 1/4 uncertainty bound also tight for discrete sequences? and are sampled Gaussians the minimizers of the uncertainty in the discrete domain? Answers to these questions will be apparent if we can solve (7).

Definition 3. We call the solution of (7) a *maximally compact* sequence.

Framing the design of maximally compact sequences as an optimization problem, we show that contrary to the continuous case, it is not possible to reach a constant time–frequency lower bound for arbitrary time or frequency spreads. We further develop a simple optimization framework to find maximally compact sequences in the time domain for a given frequency spread. In other words, we provide in a constructive and numerical way, a *sharp uncertainty principle for sequences*, later shown in Fig. 6. We also show that the Fourier spectra of maximally compact sequences are in fact a very special class of Mathieu’s functions. Using the asymptotical expansion of these functions, we develop closed-form bounds on the time–frequency spread of maximally compact sequences.

1.3. Related work

The classic uncertainty principle [13] assumes continuous-time non-periodic signals. Several works in the signal processing community also address the discrete-time/discrete-frequency case [19,6,24,20,15,11]. Our work bridges these two cases by considering the discrete-time/continuous-frequency regime.

Note that not all studies about the uncertainty principle concern the notion of spread. For example, the authors in [24] propose the uncertainty bound on the information content of signals (entropy) and [6] provides a bound on the non-zero coefficients of discrete-time sequences and their discrete Fourier transforms.

The discrete-time/continuous-frequency scenario has been recently encountered in many practical applications in signal processing. Examples include uncertainty principle on graphs [1], on spheres [14], and on Riemannian manifolds [8]. Studies on the periodic frequency spread can be found in [4] and [35]. The most comprehensive work on the uncertainty relations for discrete sequences is found in [26]. The authors show that 1/4 is a lower-bound on the time–frequency spread, which can only be achieved asymptotically as the sequence spreads in time. We provide sharp achievable bounds in the non-extreme case which match the results in [26] in the asymptotic regime.

This problem is similar to—although different than—the design of Slepian’s Discrete Prolate Spheroidal Sequences (DPSS’s). First introduced by Slepian in 1978 [30], DPSS’s are sequences designed to be both limited in the time and band-limited in the frequency domains. For a finite length, N in time and a cut-off frequency W , the DPSS’s are a collection of N discrete-time sequences that are strictly band-limited to the digital frequency range $|f| < W$, yet highly concentrated in time to the index range $n = 0, 1, \dots, N - 1$. Such sequences can be found using an algorithm similar to the Papoulis–Gerchberg method. Note the difference of such sequences to the ones that we intend to design in our work; we do not impose any constraints on the bandwidth of the sequences in the frequency domain. Also, the ideas presented here are applicable both to finite and infinite length sequences. Moreover, we focus on the concentration of the signals in the time and frequency domain using the notion of variance.

2. Main results

The following theorem is the core of the results presented in this paper.

Theorem 1. *For finding unit norm maximally compact sequences, it is sufficient to solve the following semi-definite program (SDP)*

$$\begin{aligned} & \underset{\mathbf{X}}{\text{minimize}} && \text{tr}(\mathbf{T}\mathbf{X}) \\ & \text{subject to} && \text{tr}(\mathbf{J}_0\mathbf{X}) = \alpha \\ & && \text{tr}(\mathbf{X}) = 1, \quad \mathbf{X} \succeq 0, \end{aligned} \tag{8}$$

where $\alpha = \frac{1}{\sqrt{1+\sigma^2}}$ with σ^2 the fixed periodic frequency spread. Further, \mathbf{X}^{opt} , the solution to (8) has rank one and $\mathbf{X}^{\text{opt}} = \mathbf{x}^{\text{opt}} \mathbf{x}^{\text{opt}T}$, with \mathbf{x}^{opt} the solution of (7). Matrices \mathbf{T} and \mathbf{J}_0 are defined as

$$\mathbf{T} = \begin{bmatrix} \ddots & & & & & & & \mathbf{0} \\ & 2^2 & & & & & & \\ & & 1^2 & & & & & \\ & & & 0 & & & & \\ & & & & 1^2 & & & \\ & & & & & 2^2 & & \\ \mathbf{0} & & & & & & \ddots & \end{bmatrix}, \quad \mathbf{J}_0 = \begin{bmatrix} \ddots & \ddots & & & & & & \mathbf{0} \\ & \ddots & 0 & \frac{1}{2} & & & & \\ & & \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & & & \frac{1}{2} & 0 & \ddots & & \\ \mathbf{0} & & & & \ddots & \ddots & \ddots & \end{bmatrix}. \tag{9}$$

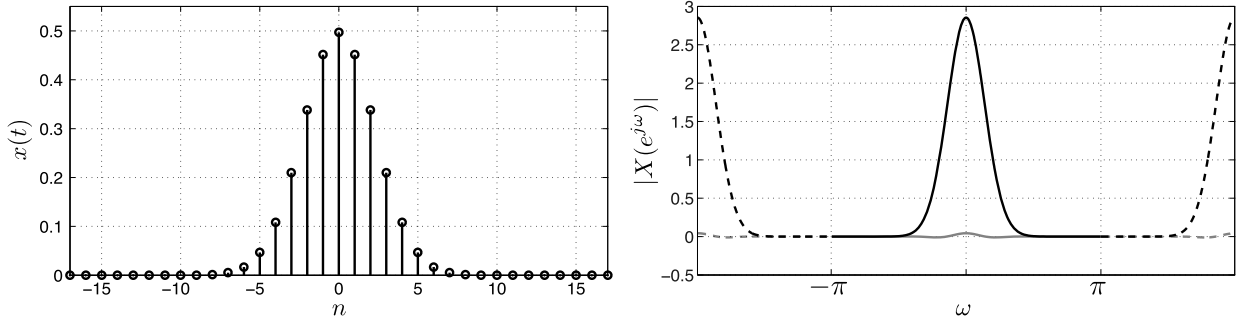


Fig. 3. An example solution of (8). The output of the SDP in (8) with $\sigma^2 = 0.1$ using cvx in Example 2. The gray curve in the frequency domain shows the difference from a fitted periodic Gaussian signal. The optimal value for Δ_n^2 is found to be 2.62 which results in a time–frequency spread of $\eta_p = 0.262$.

Proof. See Section 3.2. \square

Remark 1. The SDP (8) is feasible; indeed we can always find a signal \mathbf{x} with a certain frequency spread and norm one. Using this signal we can construct $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ which shows the feasibility of this problem.

The SDP in (8) can be solved to an arbitrary precision by using existing approaches in the optimization literature; for example using the cvx software package [10]. This gives a constructive way to design sequences that are maximally compact in the time domain with a given frequency spread.

Example 2. Take $\sigma^2 = 0.1$ to be the fixed and given frequency spread of the sequence. We can use cvx [10] to solve the semi-definite program (8) and find the optimal value of $\Delta_n^2 = 2.62$. This results in the time–frequency spread of $\eta_p = 0.262$. The simple code in MATLAB is:

```
cvx_begin
variable X(n,n);
minimize(trace(T*X))
subject to
trace(J0*X) == 1/sqrt(1+0.1)
trace(X) == 1
X == semi-definite(n)
cvx_end;
```

Note that contrary to continuous-time signals, we cannot reach the 0.25 lower bound for sequences. The resulting sequence and its DTFT are shown in Fig. 3.

Note that the dual of the SDP (8) is [5, p. 265]:

$$\begin{aligned} & \underset{\lambda_1, \lambda_2}{\text{maximize}} && \alpha\lambda_1 + \lambda_2 \\ & \text{subject to} && \mathbf{T} - \lambda_1\mathbf{J}_0 - \lambda_2\mathbf{I} \succeq 0. \end{aligned} \tag{10}$$

We will use the formulation of the dual problem many times in the rest of the paper.

Lemma 1. For maximally compact sequences, $\Delta_n^2(x)$ changes monotonically with $\Delta_{\omega_p}^2(x)$.

Proof. The feasible region of the dual (10) is shown in Fig. 4. We can write (10) as

$$\begin{aligned} & \underset{\lambda_1, \lambda_2}{\text{maximize}} && c \\ & \text{subject to} && \lambda_2 = c - \alpha\lambda_1, \\ & && \mathbf{T} - \lambda_1\mathbf{J}_0 - \lambda_2\mathbf{I} \succeq 0. \end{aligned} \tag{11}$$

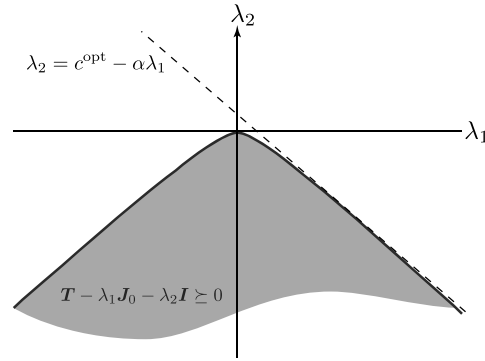


Fig. 4. The feasible set of the dual problem (10) and the supporting line. As α increases, we need to elevate the line more to support the feasible set, which means that the optimal value of Δ_n^2 increases.

Note that α changes between 0 and 1 (see Fig. 4). For a fixed α , the maximum c^{opt} is found by elevating the corresponding line $\lambda_2 = c - \alpha\lambda_1$ until it supports the feasible set (it is tangent to it). Since the feasible set is convex, as α grows (which means $\Delta_{\omega_p}^2$ decreases), we need a higher elevation of the line to support the convex set, thus c^{opt} (equivalently Δ_n^2) increases. This phenomenon is later confirmed by the simulation results in Fig. 6. □

Although Theorem 1 provides a constructive way for finding maximally compact sequences, it does not specify the closed form for these sequences. One would be interested to see if—in analogy to continuous-time—sampled Gaussians are maximally compact? The answer is negative, as shown by the following theorem:

Theorem 2. *The DTFT spectra, $X(e^{j\omega})$ of maximally compact sequences are Mathieu’s functions. More specifically,*

$$X(e^{j\omega}) = \gamma_0 \cdot \text{ce}_0(-2\lambda_1; (\omega - \omega_0)/2) e^{j\mu\omega}, \tag{12}$$

where $|\gamma_0| = \|\text{ce}_0(-2\lambda_1; (\omega - \omega_0)/2)\|^{-1}$, ω_0 and μ are shifts in frequency or time and λ_1 is the optimal solution of the dual problem (10). $\text{ce}_0(q; \omega)$ is Mathieu’s harmonic cosine function of order zero.

Mathieu’s functions—widely used in quantum mechanics [33]—are the solutions to Mathieu’s differential equation [2, §20.1.1]:

$$\frac{\partial^2 y(\omega)}{\partial \omega^2} + (a - 2q \cos(2\omega)) \cdot y(\omega) = 0. \tag{13}$$

These functions assume an even (Mathieu’s cosine function) and odd form (Mathieu’s sine function). For some specific pairs (a, q) , Mathieu’s functions can be restricted to be 2π periodic. Mathieu’s harmonic Cosine functions of order m are thus:

$$\text{ce}_m(q; \omega) = \text{ce}(a_m(q), q; \omega), \quad m \in \mathbb{N}. \tag{14}$$

For the proof of Theorem 2 and further insights on Mathieu’s functions, we refer the reader to Section 3.3.

Using the constructive method presented in Theorem 1, we can find the achievable (and tight) uncertainty principle bound for discrete sequences. This is shown and discussed more in Section 5 and Fig. 6. However, a numerically computed boundary may not always be practical, and even though the numerical solution exactly solves the problem, its accuracy may be challenged. Therefore, we characterize the asymptotic behavior of the time–frequency bound:

Theorem 3. *If x_n is maximally compact for a given $\Delta_{\omega_p}^2 = \sigma^2$, then*

$$\eta_p = \Delta_n^2 \Delta_{\omega_p}^2 \geq \sigma^2 \left(1 - \sqrt{\frac{\sigma^2}{1 + \sigma^2}} \right). \tag{15}$$

Further, for small values of σ^2 , maximally compact sequences satisfy

$$\eta_p = \Delta_n^2 \Delta_{\omega_p}^2 \leq \frac{\sigma^2}{8} \left(\frac{\sqrt{1 + \sigma^2}}{\sqrt{1 + \sigma^2} - 1} - \frac{1}{2} \right). \tag{16}$$

Proof. The proof for this theorem is provided in Section 4. \square

This fundamental result states that for a given frequency spread, we cannot design sequences which achieve the classic Heisenberg uncertainty bound. We will see how this curve compares to the classic Heisenberg bound in Section 5.

The lower bound in (15) converges to 1/2 as the value of σ^2 grows, and “pushes up” the time–frequency spread of maximally compact sequences towards 1/2 which is also an asymptotic upper bound on the time–frequency spread as $\Delta_{\omega_p}^2 \rightarrow \infty$; indeed, one may construct the unit-norm sequence $x_n^{(\varepsilon)} = \varepsilon \delta_{n+1} + \sqrt{1 - 2\varepsilon^2} \delta_n + \varepsilon \delta_{n-1}$, which verifies $\lim_{\varepsilon \rightarrow 0} \eta_p(x^{(\varepsilon)}) = 1/2$.

On the other hand, for small values of σ^2 , the upper bound in (16) converges from above to 1/4, thus “pushing down” the time–frequency spread of maximally compact sequences towards the Heisenberg uncertainty bound 1/4 from above.

2.1. Finite-length sequences

The theory that we have provided so far holds for infinite sequences. For computational purposes, we have to assume finite length for the sequences in the time domain, which is not an issue if the sequence length is chosen to be long enough. As a side benefit, a length constraint on the sequence may be added without changing the design algorithm.

3. Proof of Theorems 1 and 2

Let us start with some properties of maximally compact sequences.

3.1. Properties of maximally compact sequences

In the definitions of time and frequency spreads in Table 3 we considered complex sequences and their DTFTs. In the following, we establish two lemmas that make the search for maximally compact sequences easier. In the first lemma (Lemma 2), we state that if someone gives us a complex sequence with a given time–frequency spread, we can always take the modulus and have a sequence with a smaller or equal time–frequency spread. This enables us to only consider *real* sequences as maximally compact sequences. In Lemma 3, we state that shifts in time do not affect the time–frequency spread of sequences. This lemma allows us to assume—without loss of generality—that the sequences are centered around zero in time.

Lemma 2. *For any given sequence x_k ,*

$$\eta_p(|x|) \leq \eta_p(x).$$

Proof. Let x_k be a sequence with a given time spread $\Delta_n^2(x)$. Obviously,

$$\Delta_n^2(x) = \Delta_n^2(|x|).$$

Moreover,

$$\begin{aligned} \Delta_{\omega_p}^2(|x|) &= \left| \sum_{k \in \mathbb{Z}} |x_k| |x_{k+1}| \right|^{-2} - 1 \\ &\leq \left| \sum_{k \in \mathbb{Z}} x_k x_{k+1}^* \right|^{-2} - 1 = \Delta_{\omega_p}^2(x). \quad \square \end{aligned}$$

Recall from Lemma 1 that for maximally compact sequences Δ_n^2 changes monotonically with $\Delta_{\omega_p}^2$. Thus, it is equivalent to fix either one of Δ_n^2 or $\Delta_{\omega_p}^2$, and optimize with respect to the other.

Lemma 3. *If x is a maximally compact sequence, then $x_{k-\mu_n(x)}$ is also maximally compact. For non-integer μ_n , $x_{k-\mu_n}$ is a shorthand for since resampling on a grid shifted by μ_n in the time domain.*

Proof. As μ_n is not necessarily an integer, we can use the Parseval’s equality to show that the time center and spread is not affected by arbitrary shifts. These easy computations are left to the interested reader (they can be found in [22]). Thus, if x is a maximally compact sequence, then $x_{k-\mu_n(x)}$ is also maximally compact (note that time shift does not change the frequency characteristics of the sequence). \square

Remark 2. Using Lemmas 2 and 3, we only consider real sequences x , with $\mu_n(x) = 0$ and $\|x\|_2 = 1$. Later in Theorem 2 we show that maximally compact sequences are Mathieu’s cosine functions, which have strictly positive inverse Fourier transforms. This enables us to state that maximally compact sequences are real sequences up to a shift, scale or modulation.

3.2. Proof of Theorem 1

By using Lemmas 2 and 3, we can write problem (7) as

$$\begin{aligned} \Delta_{n,\text{opt}}^2 &= \underset{x_n}{\text{minimize}} && \sum_{k \in \mathbb{Z}} k^2 x_k^2 \\ &\text{subject to} && \sum_{k \in \mathbb{Z}} x_k x_{k+1} = \frac{1}{\sqrt{1 + \sigma^2}}, \\ &&& \sum_{k \in \mathbb{Z}} x_k^2 = 1. \end{aligned} \tag{17}$$

We can rewrite (17) in matrix form as a quadratically constrained quadratic program (QCQP) [5, p. 152]:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} & \quad \mathbf{x}^T \mathbf{T} \mathbf{x} && \underset{\mathbf{x}}{\text{minimize}} & \quad \text{tr}(\mathbf{T} \mathbf{x} \mathbf{x}^T) \\ \text{subject to} & \quad \mathbf{x}^T \mathbf{J}_0 \mathbf{x} = \alpha, & \equiv & \text{subject to} & \quad \text{tr}(\mathbf{J}_0 \mathbf{x} \mathbf{x}^T) = \alpha \\ & \quad \mathbf{x}^T \mathbf{x} = 1, & & & \quad \text{tr}(\mathbf{x} \mathbf{x}^T) = 1, \end{aligned} \tag{18}$$

where \mathbf{T} and \mathbf{J}_0 are defined in (9) and $\alpha = 1/\sqrt{1 + \sigma^2}$. Replacing $\mathbf{x} \mathbf{x}^T$ by \mathbf{X} , we can write equivalently

$$\begin{aligned}
 & \underset{\mathbf{X}}{\text{minimize}} && \text{tr}(\mathbf{T}\mathbf{X}) \\
 & \text{subject to} && \text{tr}(\mathbf{J}_0\mathbf{X}) = \alpha \\
 & && \text{tr}(\mathbf{X}) = 1 \\
 & && \mathbf{X} \succeq 0, \quad \text{rank}(\mathbf{X}) = 1.
 \end{aligned} \tag{19}$$

We further relax the above formulation to reach the semi-definite program

$$\begin{aligned}
 & \underset{\mathbf{X}}{\text{minimize}} && \text{tr}(\mathbf{T}\mathbf{X}) \\
 & \text{subject to} && \text{tr}(\mathbf{J}_0\mathbf{X}) = \alpha \\
 & && \text{tr}(\mathbf{X}) = 1, \quad \mathbf{X} \succeq 0.
 \end{aligned} \tag{20}$$

In Lemma 4 we show that the semi-definite relaxation is tight. This finishes the proof for Theorem 1. \square

Lemma 4. *The semi-definite relaxation (SDR) from (19) to (20) is tight.*

Proof. Shapiro and then Barnivok and Pataki [29,3,23,16] show that if the SDP in (19) is feasible, then

$$\text{rank}(\mathbf{X}^{\text{opt}}) \leq \lfloor (\sqrt{8m+1} - 1)/2 \rfloor, \tag{21}$$

where m is the number of constraints of the SDP and \mathbf{X}^{opt} is its optimal solution. For our semi-definite program in (20), $m = 2$. Thus, (21) implies that the solution has rank 1. Using this fact, one can see that the semi-definite relaxation is in fact tight. Recall from Remark 1 that problem (8) (and also (19)) is feasible. \square

3.3. Proof of Theorem 2

We start by problem (8) and its dual (10).

Lemma 5. *For the primal problem (8) and the dual (10), strong duality holds.*

Proof. For a semi-definite program and its dual, if the primal is feasible and the dual is strictly feasible, then strong duality holds [31,32].

We saw in Remark 1 that the primal problem (8) is feasible. For the dual, one can use the Gershgorin’s circle theorem and show that a sufficient condition for $\mathbf{T} - \lambda_1\mathbf{J}_0 - \lambda_2\mathbf{I} \succ 0$ to hold is $\lambda_2 < -\lambda_1$ and $\lambda_1 > 0$. Thus, the dual problem is strictly feasible. \square

Thus, for finding the time–frequency spread of maximally compact sequences, solving the dual problem suffices. If a sequence is a solution to the dual SDP problem (10), the dual constraint is active. Therefore, maximally compact sequences lie on the boundary of the quadratic cone

$$\mathbf{T} - \lambda_1\mathbf{J}_0 - \lambda_2\mathbf{I} \succeq 0.$$

A maximally compact sequence \mathbf{x} is thus solution of the eigenvalue problem

$$(\mathbf{T} - \lambda_1\mathbf{J}_0)\mathbf{x} = \lambda_2\mathbf{x}, \tag{22}$$

where λ_1 and λ_2 are the dual variables of the SDP problem. λ_2 is also the minimal eigenvalue of $\mathbf{T} - \lambda_1\mathbf{J}_0$ with \mathbf{x} the associated eigenvector (this can be also seen by forcing the derivative of the Lagrangian in (18) to zero).

This explicit link between the dual variables and the sequence, yields a differential equation for which the DTFT spectrum of maximally compact sequences is the solution. In the DTFT domain (22) becomes (expanding the matrix multiplications)

$$X''(e^{j\omega}) + (\lambda_2 + \lambda_1 \cos(\omega))X(e^{j\omega}) = 0, \tag{23}$$

which is Mathieu’s differential equation (13). Taking into account the periodicity of (23), it appears that not all pairs of parameters (a, q) will lead to a periodic solution. Mathieu’s functions can be restricted to be 2π periodic. The solutions of Mathieu’s harmonic differential equation—Eq. (13) with a 2π -periodic solution y —are defined as

$$\text{Mathieu’s harmonic Cosine (even, periodic)} \quad ce_m(q; \omega) = ce(a_m(q), q; \omega), \quad m \in \mathbb{N}. \tag{24}$$

$$\text{Mathieu’s harmonic Sine (odd, periodic)} \quad se_m(q; \omega) = ce(b_m(q), q; \omega), \quad m \in \mathbb{N}^+. \tag{25}$$

It is immediately visible that the spectrum of maximally compact sequences may only have the form

$$X(e^{j\omega}) = \begin{cases} \gamma_0 ce_m(-2\lambda_1; \omega/2) + \gamma_1 se_m(-2\lambda_1; \omega/2) & \text{for } m \in \mathbb{N}^+, \\ \gamma_0 ce_m(-2\lambda_1; \omega/2) & \text{for } m = 0, \end{cases} \tag{26}$$

for any constants γ_0 and γ_1 such that $\|X(e^{j\omega})\| = 2\pi$. More specifically, for any $\lambda_1 \geq 0$, the dual SDP problem can be posed and any solution would have the form (26).

Characteristic numbers of Mathieu’s equation are ordered [7, p. 113], such that for $\lambda_1 > 0$,

$$a_0(-2\lambda_1) < a_1(-2\lambda_1) < b_1(-2\lambda_1) < b_2(-2\lambda_1) < a_2(-2\lambda_1) < \dots$$

By (13) and with the substitution $\omega \rightarrow \omega/2$ one obtains $a_m(-2\lambda_1) = 4\lambda_2$. Because λ_2 is the minimal eigenvalue, we conclude that $m = 0$.

Note that this result validates the one in [12] which stated that asymptotically Mathieu’s functions minimize the time–frequency product. With a different approach, we can establish that only Mathieu’s harmonic cosine of order 0 minimizes this product for any given frequency-spread.

4. Proof of Theorem 3

Let us start by proving the lower bound.

4.1. Lower bound (15)

In order to prove the lower bound (15) we first provide the following two lemmas.

Lemma 6. Consider the sequence a_k as follows with $\theta, \nu > 0$

$$a_{k+1} = (k + 1)^2 + \theta - \frac{\nu}{a_k}. \tag{27}$$

The sequence a_k is positive for $k \geq k_0$ as soon as $a_{k_0+1} \geq a_{k_0} > 0$.

Proof. Note that

$$a_{k+2} - a_{k+1} = 3 + 2k + \nu \frac{a_{k+1} - a_k}{a_{k+1}a_k}. \tag{28}$$

Thus, by induction, the sequence is positive as soon as $a_{k_0+1} \geq a_{k_0} > 0$ for some k_0 . \square

Lemma 7. *If*

$$\lambda_2 < 1 - \sqrt{1 + \lambda_1^2}, \tag{29}$$

then $\mathbf{P} = \mathbf{T} - \lambda_1 \mathbf{J}_0 - \lambda_2 \mathbf{I} \succeq 0$.

Proof. Note that if $\lambda_1 = 0$, then the matrix \mathbf{P} is positive-definite with the given condition. In the proof we will thus assume that $\lambda_1 \neq 0$.

Consider the following tri-diagonal matrices \mathbf{P}_1 and \mathbf{P}_2 :

$$\mathbf{P}_1 = \begin{bmatrix} 1 - \lambda_2 & -\lambda_1/2 & & \mathbf{0} \\ -\lambda_1/2 & 4 - \lambda_2 & -\lambda_1/2 & \\ & -\lambda_1/2 & 9 - \lambda_2 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 0 - \lambda_2 & -\lambda_1 & & \mathbf{0} \\ -\lambda_1/2 & 1 - \lambda_2 & -\lambda_1/2 & \\ & -\lambda_1/2 & 4 - \lambda_2 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}. \tag{30}$$

Call \mathcal{I} the set of eigenvalues of \mathbf{P} , \mathcal{I}_1 the set of eigenvalues of \mathbf{P}_1 and \mathcal{I}_2 the set of eigenvalues of \mathbf{P}_2 . It is trivial to see that $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$.

We show that if condition (29) is satisfied, then both \mathbf{P}_1 and \mathbf{P}_2 have positive eigenvalues.

1. \mathbf{P}_1 : Sylvester’s criterion states that a symmetric matrix is positive definite if and only if its principal minors are all positive.¹ As \mathbf{P}_1 is symmetric, we can use Sylvester’s criterion on it. We use Gaussian elimination on the matrix \mathbf{P}_1 to compute its principal minors

$$\mathbf{P}_1^U = \begin{bmatrix} 1 - \lambda_2 & -\lambda_1/2 & & \mathbf{0} \\ 0 & s_2 & -\lambda_1/2 & \\ & 0 & s_3 & -\lambda_1/2 \\ \mathbf{0} & & 0 & \ddots & \ddots \end{bmatrix}, \tag{31}$$

where

$$s_{k+1} = (k + 1)^2 - \lambda_2 - \frac{\lambda_1^2}{4s_k}, \quad k \geq 1.$$

This satisfies the induction formula in (27). Thus according to Lemma 6, the principal minors of \mathbf{P}_1^U (and so \mathbf{P}_1) are positive as soon as $s_2 \geq s_1 > 0$. This is equivalent to $1 - \lambda_2 > 0$ and $4 - \lambda_2 - \frac{\lambda_1^2}{4(1-\lambda_2)} \geq 1 - \lambda_2$, i.e., $\lambda_1^2 \leq 12(1 - \lambda_2)$, which is a weaker condition than (29). Therefore, \mathbf{P}_1 is also positive semi-definite.

2. \mathbf{P}_2 : We can decompose \mathbf{P}_2 as

$$\mathbf{P}_2 = \begin{bmatrix} 2 & & \mathbf{0} \\ & 1 & \\ & & 1 \\ \mathbf{0} & & & \ddots \end{bmatrix} \times \begin{bmatrix} 0 - \lambda_2/2 & -\lambda_1/2 & & \mathbf{0} \\ -\lambda_1/2 & 1 - \lambda_2 & -\lambda_1/2 & \\ & -\lambda_1/2 & 4 - \lambda_2 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} = \mathbf{D} \times \mathbf{P}_2^{(s)}. \tag{32}$$

Note that both \mathbf{D} and $\mathbf{P}_2^{(s)}$ are symmetric. Also observe that the eigenvalues of \mathbf{P}_2 and $\mathbf{P}_2^{(s)}$ are equal. Thus, it suffices to consider the eigenvalues of $\mathbf{P}_2^{(s)}$; If $\mathbf{P}_2^{(s)}$ is positive-definite then all the eigenvalues of \mathbf{P}_2 are positive.

¹ For an infinite matrix that can be regarded as the matrix of a bounded operator in ℓ^2 , one can show that Sylvester’s criterion still holds, i.e., the quadratic form is non-negative definite if and only if all the finite principal minors of the matrix are positive. Note that we consider $\|x\|_2 = 1$, thus $x \in \ell^2$.

Again using Gaussian elimination on $\mathbf{P}_2^{(s)}$ results in

$$\mathbf{P}_2^{(s),U} = \begin{bmatrix} -\lambda_2/2 & -\lambda_1/2 & 0 & & \mathbf{0} \\ 0 & s_1 & -\lambda_1/2 & & \\ & 0 & s_2 & -\lambda_1/2 & \\ \mathbf{0} & & 0 & \ddots & \ddots \end{bmatrix}, \tag{33}$$

where $s_1 = 1 - \lambda_2 + \lambda_1^2/2\lambda_2$ and s_k has the following form $s_{k+1} = (k + 1)^2 - \lambda_2 - \frac{\lambda_1^2}{4s_k}$, $k \geq 1$. This again satisfies the induction formula (27). Therefore, it suffices to show that $\lambda_2 < 0$ and $s_2 \geq s_1 > 0$. We show that under condition (29), this is true. In order to satisfy $s_1 > 0$, we need $1 - \lambda_2 + \lambda_1^2/2\lambda_2 > 0$, which is equivalent to $\lambda_2 < \frac{1}{2}(1 - \sqrt{1 + 2\lambda_2^2})$. This is a weaker condition than (29).

Further, in order to satisfy $s_2 > s_1$, we need

$$s_2 - s_1 = 3 + \frac{\lambda_1^2}{4s_1\lambda_2^2}(\lambda_2^2 - 2\lambda_2 - \lambda_1^2) \geq 0.$$

Thus, it is enough to have $\lambda_2^2 - 2\lambda_2 - \lambda_1^2 \geq 0$. That is $\lambda_2 \leq 1 - \sqrt{1 + \lambda_1^2}$, which is the bound provided in (29). Putting these together, we can conclude that under condition (29), the matrix \mathbf{P} is positive-definite. \square

Note that condition (29) gives a sufficient (but not necessary) condition on the feasible set of the dual problem (10). Thus, it provides a lower bound for the maximum value of the dual. Consider the restricted dual problem

$$\begin{aligned} &\underset{\lambda_1, \lambda_2}{\text{maximize}} && \alpha\lambda_1 + \lambda_2 \\ &\text{subject to} && \lambda_2 < 1 - \sqrt{1 + \lambda_1^2} \end{aligned} \tag{34}$$

The solution to this problem is simply $1 - \sqrt{1 - \alpha^2}$. If we rewrite α in terms of σ^2 , we finally have

$$\Delta_{n,\text{opt}}^2 \geq 1 - \sqrt{\frac{\sigma^2}{1 + \sigma^2}}.$$

This concludes the proof for the lower bound.

4.2. Upper bound (16)

It is easy to see that for small values of σ^2 (equivalently α closer to 1), the maximum of the dual problem is achieved for large values of λ_1 (remember that λ_1 needs to be positive). We saw in 3.3 that for maximally compact sequences (i.e. sequences that result in the maximum of the dual problem) we have $\lambda_2 = 1/4 a_0(2\lambda_1)$ (note that $a_0(-q) = a_0(q)$). McLachlan in [17] shows that for large enough values of q , we have

$$\begin{aligned} a_0(q) &= -2q + 2q^{\frac{1}{2}} - \frac{1}{4} - \frac{1}{32}q^{-\frac{1}{2}} - \frac{48}{2^7}q^{-1} - \frac{848}{2^{17}}q^{-\frac{3}{2}} - \frac{4,752}{2^{20}}q^{-2} - \frac{126,752}{2^{20}}q^{-\frac{5}{2}} - \dots \\ &= -2q + 2q^{\frac{1}{2}} - \frac{1}{4} - \frac{1}{32}q^{-\frac{1}{2}} + O(q^{-1}). \end{aligned} \tag{35}$$

Thus, we have for large q ,

$$a_0(q) \leq -2q + 2q^{\frac{1}{2}} - \frac{1}{4}. \tag{36}$$

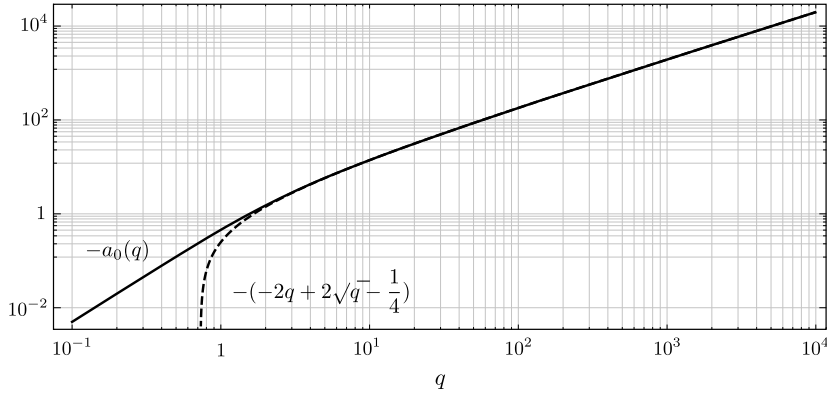


Fig. 5. The characteristic number $a_0(q)$ of Mathieu’s function together with its upper bound. In order to plot on a log scale, the negations are plotted.

The values for $a_0(q)$ and its upper bound are shown in Fig. 5. In order to be able to plot on a log scale, their negative counterparts are plotted.

After replacing q by $2\lambda_1$ and $a_0(2\lambda)$ by $4\lambda_2$ in (36), we have

$$\lambda_2 = \frac{1}{4}a_0(2\lambda_1) \leq -\lambda_1 + \frac{1}{\sqrt{2}}\sqrt{\lambda_1} - \frac{1}{16}. \tag{37}$$

Because this set contains the original feasible set of the dual problem, it will give an upper bound on the optimal value of the dual. In other words, $\Delta_n^2 \leq \sigma_r^2$, where

$$\begin{aligned} \sigma_r^2 &= \text{maximize}_{\lambda_1, \lambda_2} \quad \alpha\lambda_1 + \lambda_2 \\ &\text{subject to} \quad \lambda_2 \leq -\lambda_1 + \frac{1}{\sqrt{2}}\sqrt{\lambda_1} - \frac{1}{16}. \end{aligned} \tag{38}$$

It is easy to see that the maximum of (38) is achieved for $\lambda_1 = \frac{1}{8(1-\alpha)^2}$. Replacing λ_1 in (38) and using the fact that $\Delta_{n,\text{opt}}^2 \leq \sigma_r^2$, leads to

$$\Delta_{n,\text{opt}}^2 \leq \frac{1}{8} \left(\frac{\sqrt{1+\sigma^2}}{\sqrt{1+\sigma^2}-1} - \frac{1}{2} \right). \tag{39}$$

5. Simulation results

In order to show the behavior of the results obtained in Theorems 1 and 3, we ran some simulations. For this, we assumed that the designed filter is finite length with 201 taps in the time domain. The length is long enough not to pose restrictions on the solution for the considered frequency spreads. For smaller frequency spreads, we can increase the length of the sequence. For different values of $\Delta_{\omega_p}^2 = \sigma^2$, we solved the semi-definite program (8) using the cvx toolbox in MATLAB.

The resulting values of Δ_n^2 were then multiplied with the corresponding $\Delta_{\omega_p}^2$ to produce the time–frequency spread of maximally compact sequences. The time–frequency spread of maximally compact sequences versus their frequency spread is shown with the solid curve in Fig. 6. This means—numerically—that any time–frequency spread under this curve is not achievable. The dotted line in this figure shows the classic Heisenberg uncertainty bound. Comparing the two curves shows the gap between the classic Heisenberg principle and what is achievable in practice. The dashed lines represent analytical lower and upper bounds for the time–frequency spread of maximally compact sequences (found in Theorem 3).

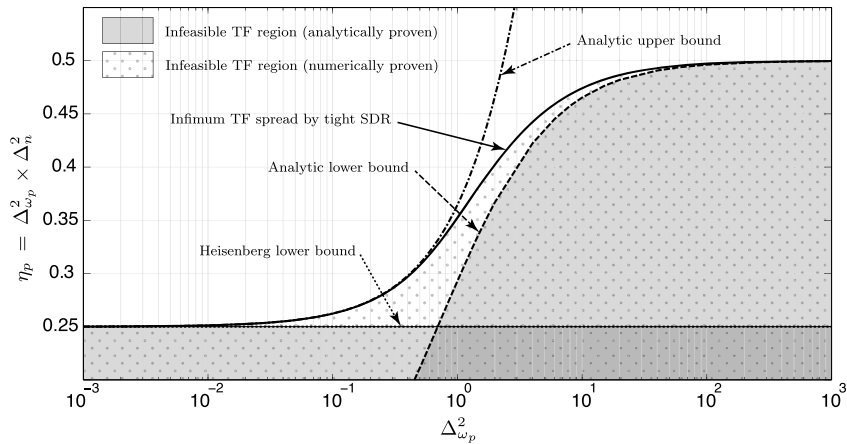


Fig. 6. New uncertainty bounds. The solid line shows the results of solving the SDP in (8). The dotted line shows the classic Heisenberg uncertainty principle. The dashed lines show the analytic lower and upper bounds found in Theorem 3.

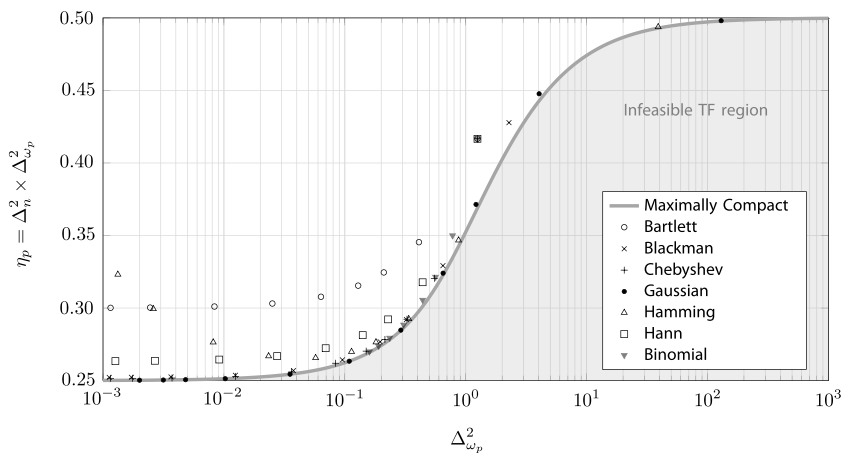


Fig. 7. Time–frequency spread of common FIR filters. By changing the length of the filters in time, we compute the time and frequency spreads for each type of filters. For small values of the frequency spread, Gaussian filters are good approximations of Mathieu’s functions (as shown also in [26]).

Further, to give an insight on how the time–frequency spread of some common filters compare to that of maximally compact sequences, we plot their time–frequency spread together with the new uncertainty bound in Fig. 7. By changing the length of each filter in time, we can find its time and frequency spreads which results in a point on the figure. We observe that as shown by Prestin et al. in [26], asymptotically when the frequency spread of sequences are very small, sampled Gaussians converge to the lower bound for maximally compact sequences.

6. Conclusion

We showed that for discrete-time sequences, contrary to continuous-time signals, the classic Heisenberg uncertainty bound is not always achievable and the uncertainty minimizers have a large gap from this bound. We constructed an optimization framework for finding the uncertainty minimizers which we refer to as the “maximally compact sequences”. This framework allows one to find—numerically and efficiently—the most compact sequence in time with a desired frequency spread. We further showed that the discrete-time Fourier transform of these sequences is a very special class of Mathieu’s functions. We also proved analytic bounds on the time–frequency spread of discrete sequences. Maximally compact sequences can serve as

optimal probing windows (similarly to the ones shown in Fig. 7) in several signal processing applications. Furthermore, the connection provided between the solutions of the semi-definite program and Mathieu's functions, also enables the approximation of Mathieu's functions with arbitrary accuracy.

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